stichting
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AFDELING ZUIVERE WISKUNDE ZW 33/74 NOVEMBER

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# A COMPACT TREELIKE SPACE IS THE CONTINUOUS IMAGE OF AN ORDERED CONTINUUM



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*Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.*

*The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.*

AMS (MOS) subject classification scheme (1970): 54D05

# A compact treelike space is the continuous image of an ordered continuum.

by

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ABSTRACT

The theorem stated in the title is proved.

## 1. DEFINITION AND BASIC PROPERTIES OF TREELIKE SPACES.

Two points of a topological space are called *separated* when they do not belong to the same quasicomponent.
A connected topological space X is called *treelike* when given any two points $p,q \in X$ there is a point $z \in X$ separating p and q (that is, such that p and q are separated in $X \backslash \{z\}$).
If p,q are points of a treelike space X then we define

$$E(p,q) := \{z \mid z \text{ separates } p \text{ and } q \text{ in } X\}$$
$$S(p,q) := \{p\} \cup E(p,q) \cup \{q\}.$$

$E(p,q)$ is empty if and only if $p = q$.

We will write $X\backslash z = A_p + B_{q,r}$ meaning that $X\backslash\{z\}$ is the topological sum of its subspaces A and B, while $p \in A$ and $\{q,r\} \subset B$.

<u>LEMMA 1.1</u>. *Let* X *be connected,* C *a connected subspace of* X. *If* $X\backslash C = A + B$ *then both* $A \cup C$ *and* $B \cup C$ *are connected.*

<u>LEMMA 1.2</u>. *Let* X *be connected,* C *a connected subspace of* X *and* S *a component of* $X\backslash C$. *Then* $X\backslash S$ *is connected.*

Slightly stronger versions of these lemma's are given in KOK [3].
Using these lemma's it is not too difficult to prove the propositions below. Most of them can be found in KOK [3].

<u>PROPOSITION 1.3</u>. *A treelike space is Hausdorff.*

<u>PROPOSITION 1.4</u>. *Let* X *be treelike. Then for all* $p \in X$ *each component of* $X\backslash p$ *is open.*

<u>PROPOSITION 1.5</u>. *Let* X *be treelike and* $p,q,r \in X$. *Then* $S(p,q) \cap S(p,r) \cap S(q,r)$ *is a singleton.*

PROOF: In BROUWER & SCHRIJVER [1] it is shown that this intersection is nonempty (see also [3] pp.45-50). On the other hand it is easily seen that it cannot contain more than one point:
Suppose $y,z \in S(p,q) \cap S(p,r) \cap S(q,r)$ , $y \notin \{p,q,r,z\}$.
Then e.g. $X\backslash y = A_{p,z} + B_q + C_r$ , but now $B \cup \{y\} \cup C$ is connected in $X\backslash z$ so that z doesn't separate q and r. The case that some or all of the points y, z, p, q, r coïncide is handled similarly. □

Let X be a treelike space and $p,q \in X$. $S(p,q)$ can be ordered in a natural way (separation order):

PROPOSITION 1.6. *The relation* $\leq$ *on* $S(p,q)$ *defined by* $x \leq y$ *iff* $x \in S(p,y)$ *defines a continuous ordering on* $S(p,q)$ *(i.e. an ordering without jumps or gaps).*

PROPOSITION 1.7.

(i) $S(p,q)$ *is closed.*

(ii) *If* $S(p,q)$ *is connected and locally connected, then it is compact.*

In general $S(p,q)$ is not connected, but we have:

PROPOSITION 1.8. *Let* X *be treelike and either locally connected* [WHYBURN] *or locally peripherally compact* [PROIZVOLOV]. *Then for all* $p,q \in X$, $S(p,q)$ *is an ordered continuum, the intersection of all connected sets containing both* p *and* q.

Intuitively clear (and an easy consequence of proposition 1.5) is the following

PROPOSITION 1.9. [H. KOK, p.44]. *Let* X *be treelike and suppose that for each* $p \in X$, $X\backslash p$ *has at most two components. Then* X *is (weakly) orderable.*

A *based treelike space* is a triple (X,a,b) where X is a treelike space and a and b are (not necessarily different) points of X. As usual we identify a based treelike space with its underlying treelike space.

Let (X,a,b) be a based treelike space. The subset S(a,b) of (X,a,b) is called its base. In a based treelike space we can define a canonical projection

$$\pi : X \to S(a,b)$$

by

$$\{\pi(x)\} = S(a,b) \cap S(a,x) \cap S(x,b).$$

(Well defined because of proposition 1.5)

If S(a,b) is not connected, then certainly $\pi$ cannot be continuous, but we have:

<u>PROPOSITION 1.10.</u>

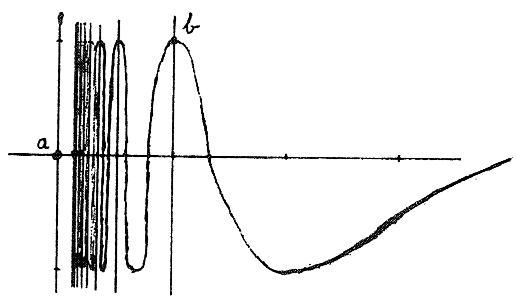
(i) $\pi \circ \pi = \pi$

(ii) *If* $c \in S(a,b)$ *and* $X \backslash c = A_a + B_b + C$ *where* A *and* B *are connected (and empty when* a = c *resp.* b = c), *then* $\pi^{-1}(c) = \bar{C} = C \cup \{c\}$.
*That is,* $\pi^{-1}(c)$ *is a closed connected set (called the stalk at* c).

(iii) *If* S(a,b) *is connected and locally connected then* $\pi$ *is continuous. In particular this is the case when* X *is locally connected or locally peripherally compact.*

<u>PROOF OF (iii)</u>: Let $U \subset S(a,b)$ be open in S(a,b). Since S(a,b) is locally connected we may suppose U to be connected: U = E(p,q) for some p,q with $a \le p < q \le b$. Now $\partial\pi^{-1}U = \{p,q\}$, so $\pi^{-1}U$ is open. □

Note: connectedness of S(a,b) alone is not sufficient:




Take $X = \{(x,y)\ \mathbb{R}^2 \mid (x>0 \wedge y=\cos\frac{2\pi}{x}) \vee (\exists k \in \mathbb{N}: x=\frac{1}{k}) \vee x=y=0\}$ with subspace topology of $\mathbb{R}^2$

a = (0,0)

b = (1,1)

Here $\pi$ is not continuous.

## 2. THE CONSTRUCTION

THEOREM *Let* X *be a compact treelike space. Then* X *is the continuous image of an ordered continuum.*
This answers the question raised by V.V. PROIZVOLOV in [4].

PROOF. We will construct compact treelike spaces $X_\alpha$ and maps $p_{\alpha\beta} : X_\alpha \twoheadrightarrow X_\beta$ $(0 \leq \beta \leq \alpha \leq \alpha_0)$ forming an inverse system, such that

$$X_{\alpha_0} = \lim_{\leftarrow} X_\alpha \text{ is an ordered continuum and } p_{\alpha_0 0} : X_{\alpha_0} \twoheadrightarrow X_0 = X$$

is the required mapping.
More in particular we will have based treelike spaces $(X_\alpha, a_\alpha, b_\alpha)$ and for all $\beta \leq \alpha$ : $p^{-1}_{\alpha\beta}(a_\beta) = \{a_\alpha\}$ , $p^{-1}_{\alpha\beta}(b_\beta) = \{b_\alpha\}$.
The idea is to construct the spaces $X_\alpha$ by transfinite induction, constantly enlarging the base, until finally $X_{\alpha_0} = S(a_{\alpha_0}, b_{\alpha_0})$. Assume that X contains more than one point.

(i) Step 0.
First set $X_0 = X$, $p_{00} = 1_X$ and choose two endpoints $a_0$ and $b_0$ from X. (This is possible because of the lemma: every nondegenerate compact connected $T_1$ space has at least two end points (see [5])).

(ii) A successor step.
Suppose $X_\beta$ and $p_{\gamma\beta}$ defined for all ordinals $\gamma \leq \beta < \alpha$.
If $\alpha$ is a successor: $\alpha = \delta + 1$, and $X_\delta$ is orderable then we're done: set $\alpha_0 = \delta$. Therefore suppose $X_\delta$ not orderable.
By proposition 1.9 we can find a ramification point p and by projecting it we may assume it to lie on $S(a_\delta, b_\delta)$, i.e. a point p such that $X_\delta \backslash p = A_{a_\delta} + B_{b_\delta} + C$ where A and B are clopen in $X_\delta \backslash p$ and C is non-empty, open and connected (use proposition 1.4).
$\bar{C} = C \cup \{p\}$ is compact and thus has an endpoint q different from p.
Define $Y_\alpha := (A \cup \{p\} \cup C) \times \{0\}$ , $Z_\alpha := (S(q,p) \cup B) \times \{1\}$,
$X_\alpha := (Y_\alpha + Z_\alpha)/R$
where R identifies $(q,0) \in Y_\alpha$ with $(q,1) \in Z_\alpha$.

Define $p_{\alpha\alpha} = 1_{X_\alpha}$ ,

and $p_{\alpha\delta}((r,i)) = r \qquad (r \in X_\delta \ , \ i=0,1)$ ;

and $p_{\alpha\beta} = p_{\delta\beta} \circ p_{\alpha\delta} \qquad (\beta<\delta)$.

Now $X_\alpha$ is compact and treelike as is easily verified.
Note that the endpoints of $X_\alpha$ are exactly those of the form $p_{\alpha\delta}^{-1}(r)$ with r endpoint of $X_\delta$ different from q.
If we define $e_\alpha := p_{\alpha 0}((q,i)) = p_{\delta 0}(q) \in X$ then we can say that in this $\alpha$-th step the endpoint $e_\alpha$ of X is removed, while the status of the other endpoints remains unchanged.

(iii) A limit step.
If $\alpha$ a limit ordinal, set $X_\alpha = \varprojlim\{X_\beta | \beta<\alpha\}$ and $p_{\alpha\beta} : X_\alpha \twoheadrightarrow X_\beta$ the canonical projection.
$X_\alpha$ is compact and connected (see e.g. Dugundji [2] p.435).

<u>Claim</u>: $X_\alpha$ is treelike.
Let $x,y \in X_\alpha$, $x \neq y$ and define $x_\beta := p_{\alpha\beta}(x)$ , $y_\beta := p_{\alpha\beta}(y)$.

If for some $\beta < \alpha$ and $r \in X_\beta$ we have $p_{\alpha\beta}^{-1}(r) = \{s\}$ is a singleton, and r separates $x_\beta$ and $y_\beta$, then s separates x and y:

$$X_\beta \setminus r = \underset{x_\beta}{A} + \underset{y_\beta}{B} \Rightarrow X_\alpha \setminus s = p_{\alpha\beta}^{-1}(X_\beta \setminus r) = \underset{x}{p_{\alpha\beta}^{-1}(A)} + \underset{y}{p_{\alpha\beta}^{-1}(B)} .$$

Therefore assume that if for some $\beta$ r separates $x_\beta$ and $y_\beta$ then $|p_{\alpha\beta}^{-1}(r)| > 1$.

If for some $\beta < \alpha$ $\pi x_\beta \neq \pi y_\beta$ then choose $w \in E(\pi x_\beta, \pi y_\beta)$.
Let $\gamma$ be the first ordinal $> \beta$ such that $|p_{\gamma\beta}^{-1}(w)| > 1$.
Then $\gamma$ is a successor, and $|p_{\gamma\beta}^{-1}(w)| = 2$.
At the $\gamma$-th step of the construction some interval S(q,w) is inserted in the base somewhere between $\pi x_\gamma$ and $\pi y_\gamma$ (Here $\{q\} = p_{\gamma-1\ 0}^{-1}(e_\gamma)$) .
But this interval (or at least E(q,w)) is left untouched during all following steps since it doesn't contain any ramification points; that is, if $r \in E(q,w)$ then $|p_{\alpha\gamma}^{-1}(r)| = 1$, a contradiction.
Therefore for all $\beta < \alpha$ we have $\pi x_\beta = \pi y_\beta$ .

Since $x \neq y$ for some $\beta < \alpha$ we have $x_\beta \neq y_\beta$ .
Let $w$ separate $x_\beta$ and $y_\beta$. Let $\gamma$ be the first ordinal $> \beta$ such that $|p_{\gamma\beta}^{-1}(w)| > 1$.
Then $p_{\gamma\beta}^{-1}(w) = \{w_1, w_2\}$, and at least one of $w_1$ and $w_2$ separates $x_\gamma$ and $y_\gamma$.
But both $w_1$ and $w_2$ lie in the base of $X_\gamma$, so we might as well assume $w \in S(a_\beta, b_\beta)$.
Now we have $w = \pi x_\beta = \pi y_\beta$.
Let $X_\beta \backslash w = P_{x_\beta} + Q_{y_\beta} + R_{a_\beta, b_\beta}$ where P and Q are connected.

<u>Claim</u>: For each $r \in P \cup Q$ $|p_{\alpha\beta}^{-1}(r)| = 1$.
For, let $\gamma$ be the first ordinal $> \beta$ such that for some $r \in P$ $|p_{\gamma\beta}^{-1}(r)| > 1$.
Then at step $\gamma$ in the construction we had $w$ (or rather $w'$ with $\{w'\} = p_{\gamma-1\ \beta}^{-1}(w)$) as a ramification point, while the endpoint $q$ chosen lies in P.
Let $\{z\} = S(w,q) \cap S(q, x_{\gamma-1}) \cap S(x_{\gamma-1}, w) \subset X_{\gamma-1}$.
Then (with obvious identifications) $\pi x_\gamma = z \neq w'' = \pi y_\gamma$ , contradiction.

From this claim it follows that $P' = p_{\alpha\beta}^{-1}[P] \subset X_\alpha$ is homeomorphic to P while $\partial P' = \{w'\}$ for some $w' \in p_{\alpha\beta}^{-1}(w)$.
Likewise $Q' = p_{\alpha\beta}^{-1}[Q]$ is mapped homeomorphically onto Q by $p_{\alpha\beta}$ while $\partial Q' = \{w''\}$. Since always $\pi x_\gamma = \pi y_\gamma$ we have $w' = w''$. But this means that $X_\alpha \backslash w' = P'_x + Q'_y + R'_{a_\alpha, b_\alpha}$ , which proves that $X_\alpha$ is indeed treelike.
Furthermore, if e is an endpoint of $X_\alpha$ then for all $\beta < \alpha$ $p_{\alpha\beta}(e)$ is an endpoint of $X_\beta$.

Since each successor step eliminates another of the endpoints of X, and a limit step doesn't introduce any new endpoints, there must be a last space $X_{\alpha_0}$ not containing any ramification points.
But this means that this $X_{\alpha_0}$ is orderable (proposition 1.9). □

## 3. AN EXAMPLE

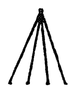
The foregoing seems to be a rather long proof for such an intuitively obvious construction. One has however to be careful since it is not generally true that the inverse limit of treelike spaces is again treelike:

Take an interval : $X_0 = C\{\frac{1}{2}\}$ , the cone over $\frac{1}{2}$ 

split it into two : $X_1 = C\{\frac{1}{4},\frac{3}{4}\}$ 

then split each half into two intervals:

$$X_2 = C\{\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\}$$



etc.

Formally:

Let $X_n = C\{\frac{k}{2^{n+1}} \mid k \text{ odd},\ 1 \le k \le 2^{n+1}\} = \{\frac{k}{2^{n+1}}\} \times I/R$

where R identifies all points (x,1).

Let $p_{nm}(\frac{k}{2^{n+1}},y) = (\frac{1}{2^{m+1}},y)$ where $|\frac{k}{2^{n+1}} - \frac{1}{2^{m+1}}| < \frac{1}{2^{m+1}}$ $(m \le n)$ .

Then $\varprojlim X_n$ is the cone over a Cantor set, and certainly not treelike (compact but not locally connected).

## REFERENCES


[1] BROUWER, A.E. and A. SCHRIJVER, *A characterization of supercompactness with an application to treelike spaces*, Mathematical Centre report ZW 34/74, Amsterdam, 1974.

[2] DUGUNDJI, J., *Topology*, Allyn and Bacon, Boston 1966.

[3] KOK, H., *Connected Orderable spaces*, MC tract 49, Amsterdam 1973.

[4] PROIZVOLOV, V.V., *On peripherally bicompact treelike spaces*. Soviet Math. Dokl. 10 (1969) no.6, pp.1491-1493.

[5] WHYBURN, G.T., *Cut points in general topological spaces*, Proc. Nat. Acad. of Sci. 61 (1968), pp.380-387.